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# ILLUSTRATING DIFFERENTIAL GEOMETRY VIA GEOMETRIC ALGEBRA OF COLOR SPACE

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**Abstract.** *Both Geometric Algebra (GA) and Differential Geometry (DG) distinguish among different kinds of vectors of the same dimensionality, leading to a higher expressiveness of algebraic objects than usual vector calculus based on Euclidean geometry. The distinct semantics of different kinds of vectors  vector, co-vector, bi-vector, bi-co-vector - is obscured in the framework of Euclidean vector calculus, but they can be reasonably associated with a specific visual representation. However, in just three dimensions various ambiguities arise which question the requirement to distinguish these vector types. Higher dimensions help to illustrate this requirement, but only in five dimensions all these four vector types become unambiguous. A spatial representation in five dimension is unfortunately hard to comprehend. In this article we consider colors as dimensions instead of spatial coordinates, leading to considerations that allow improved understanding of the differences between vectors, co-vectors, bi-vectors and bi-co-vectors. This discussion is primarily intended for didactic purposes, but may have actual applications in image processing.*

# 1 INTRODUCTION

## 1.1 Related Work

D. Hestenes [6, 7] introduced Geometric Algebra as a tool providing more insight into mathematical formalisms, in particular (but not only) the spinor formalism in quantum mechanics. Geometric Algebra applies to arbitrary dimensions and when applied to a four-dimensional spacetime allows to express special and general relativity within this framework [5]. In spacetime algebra (STA) space and time have different properties which impedes understanding of four-dimensional structures. The four-dimensional computer graphics presented by A. Hansons [4] considers a fully spatial four-dimensional situation where all coordinates are equivalent. Unfortunately such a four-dimensional space is less intuitive since it is unphysical and therefore beyond human experience. However, higher dimensional spaces are still intuitive when considering colors instead of spatial dimensions, as will be demonstrated in this article. It is not uncommon to consider colors as elements of a vector space, and such approaches may be useful for image processing [3]. In previous work [1] we discussed the properties of using the framework of geometric algebra in conjunction with differential geometry without performing the frequently done identifications between vector-valued objects. In this article we introduce texturing as additional dimension to a color space, leading to color patterns as the basic objects of consideration instead of geometric objects. A metric on color patterns as a measure of perceptual similarity has been investigated by Mojsilovic et.al. [8]; however, they did not express their findings in vision research using a mathematical framework such as the one presented here.

## 1.2 Outline

Section 2 reviews the mathematical framework introducing the notation and terminology used throughout this article. While each of the reviewed theories can be found in standard text books, they are usually presented in their own, often incompatible or sometimes even obscure notation which impacts insight. Section 2.1 reviews the basic definitions of a vector space, which is the sufficient for defining the Grassmann algebra in section 2.2. Section 2.3 reviews the definition of tangential and co-tangential vectors as well as the definition of a metric. The metric is essential for defining the geometric algebra, reviewed in section 2.4. Section 2.5 finally discusses applying the Grassmann and Geometric algebra on the tangential and co-tangential vectors. Section 3 discusses the abstract relationships introduced in section 2 in virtue of colors instead of geometrical objects: Section 3.1 discusses the three-dimensional case using the RGB color model, section 3.2 the four-dimensional case and section 3.3 presents the extension to five dimensions.

# 2 MATHEMATICAL FRAMEWORK

Differential Geometry is a most fundamental mathematical framework crucial to physics as it describes the concept of a manifold with charts. Many formalism in physics and in particular simplifications used in engineering are based on vector calculus in euclidean space, which can be seen as a special case of the more general framework. Only in certain application domains such as general relativity or computational modeling using curvilinear grids it is evident that a more powerful framework than Euclidean vector calculus is required. In such more general cases the actual structures that are implicitly assumed in Euclidean vector algebra become visible and explicit. The awareness of such otherwise implicitly used structures is helpful for understanding the mathematical framework and to ease generalizing algorithms. This section follows mostly the introduction of [1].

### 2.1 Vector Spaces

The concept of a vector space is most fundamental, as it considers abstract objects with certain constraining properties: A vector space over a field $F$ is a set $V$ together with two binary operations, called scalar multiplication and vector addition, satisfying the vector space axioms. A significant consequence is that a vector space is closed under these operations, i.e. for arbitrary $u, v \in V$ and $a, b \in F$, then it follows that $au + bv \in V$.

An element of $V$ is called a "vector", commonly illustrated via an arrow. However, the vector space elements may have a variety of other properties distinguishing them and the representation as an arrow is not always appropriate to them.

Some formalisms may be implied directly on vector spaces such as the grassmann algebra, others such as geometric algebra require additional structures, for instance a metric. The concept of a space-time, fundamental to physics, is not a vector space, though in simplifications it is treated as such. In general however the full framework of differential geometry needs to be applied, which in addition to a metric also may require more structures such as an orientation-form and possibly a connection form, as will be discussed below.

### 2.2 Grassmann Algebra

Grassmann Algebra [2] introduces another binary operation to a vector space, the Grassmann product "$\wedge$", also called the wedge, alternating or exterior product. It can be seen as the anti-symmetric subset of the direct product of two vectors $v \wedge u = v \otimes u - u \otimes v$ and provides the important property

$$v \wedge u = -v \wedge u \tag{1}$$

The Grassmann product introduces a new type of vectors, so-called bi-vectors, which form their own vector space distinct, but related to, vectors. When vectors are illustrated by an arrow, a bi-vector is appropriatly illustrated by the plane that is spun by two vectors as in Fig. 1. Bi-vectors

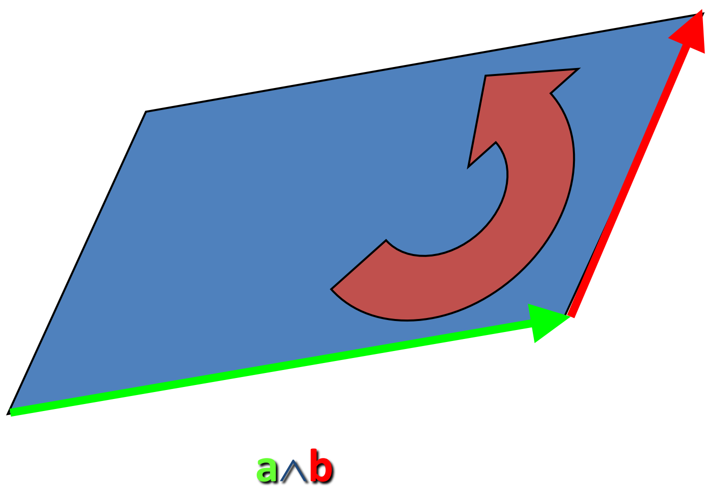


Figure 1: A bi-vector is represented by an (oriented) area element spun by its two generating vectors.

form their own vector space, called $\Lambda^2(V)$ or $V \wedge V$. With $n$ the dimensions of a vector space $V$, the dimension of $\Lambda^2(V)$ is $n(n-1)/2$.

Higher order wedge products are defined by applying the wedge product to bi-vectors and vectors etc., leading to the k-th power $\Lambda^k(V)$ of a vector space $V$. Its element is a k-vectors with $\binom{n}{k}$ components. However, since $v \wedge v = 0$ due to eqn. (1), the highest power can only be $k = n$ for an $n$-dimensional vector space as there are at most $n$ linearly independent vectors. $\Lambda^n(V)$ is a one-dimensional vector space as all elements, the so-called pseudo-scalars, are scalar multiples of the basis vector $v_1 \wedge v_2 \ldots v_n$ where $v_1, v_2 \ldots v_n$ are the $n$ basis vectors of $V$. Pseudo-scalars represent oriented ($n$-dimensionial) volumes, and - in contrast to a actual scalars - depend on the used coordinate system. For instance, they change sign under reflections.

These dimensionality of $k$-vectors are illustrated by the Pascal's triangle, with the row representing the dimensionality $n$ (starting with 0D in the first row) of the underlying vector space and the column the power $k$ of the $k$-vector:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array} \tag{2}$$

For a three-dimensional vector space, there are three vectors as well as three bi- vectors. Therefore, in three dimensions vectors and bi-vectors can be identified. Bi- vectors describing a surface area are known as "normal vector" in usual Euclidean vector calculus. For a four-dimensional vector space this identification of vectors with bi-vectors is no longer possible, as there will be four 1-vectors, six bi-vectors, four tri-vectors and one 4-vector.

### 2.3 Differential Geometry

Differential Geometry (DG) identifies derivatives along curves in a manifold as tangential vectors. Given a curve $q : \mathbb{R} \to M : s \mapsto q(s)$, the derivative $d/ds$ along the curve can be written as a linear combination of basis vectors (e.g. $\{\partial_x, \partial_y, \partial_z\}$), which each are derivatives in the direction of a (local) coordinate system (e.g. given by coordinate functions $\{x, y, z\}$):

$$d/ds = \dot{q}^x \partial_x + \dot{q}^y \partial_y + \dot{q}^z \partial_z \quad . \tag{3}$$

The numbers $\{\dot{q}^x, \dot{q}^y, \dot{q}^z\}$ represent the tangential vector $\dot{q}$ in the coordinate system $\{x, y, z\}$. The notation "$\partial_x$" hereby stands for "derivation into the direction of the coordinate function $x$" and is a shortcut for the partial derivative $\partial/\partial x$. The set of all tangential vectors at a certain point $P \in M$ is called the tangential space $T_p(M)$. The tangential space is a vector space regardless whether the underlying manifold $M$ is a vector space itself or not.

In addition to tangential vectors differential geometry considers linear functions on tangential vectors $df : T_p(M) \to \mathbb{R}$, which constitute a vector space by itself. They can be constructed from scalar functions $f : M \to \mathbb{R}$ such that for any tangential vector $v \in T_p(M)$

$$df(v) := v(f) \equiv (v^x \partial_x + v^y \partial_y + v^z \partial_z)(f) \quad . \tag{4}$$

These linear functions on tangential vectors are called co-vectors and their space is the co-tangential space $T_p^*(M)$. The basis of the co-tangential space in a coordinate system $\{x, y, z\}$ is denoted $\{dx, dy, dz\}$ following the notation of a scalar function's total differential

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \quad . \tag{5}$$

Tangential vectors and co-vectors are dual to each other, i.e. when applying the $k^{th}$ basis co-vector as $dx^k$ to the $i^{th}$ basis vector $\partial_i$ the result will be $1$ only if $i = k$, otherwise $0$. This property is intrinsic to the tangential and co-tangential spaces on an arbitrary manifold.

Additional properties that may be specified on a certain manifold are in particular the metric tensor field $g : T_p(M) \times T_p(M) \to \mathbb{R}$ that allows to define the inner product of two vectors and therefore means to compute the norm (magnitude) of tangential vectors (a co-metric $g : T_p^*(M) \times T_p^*(M) \to \mathbb{R}$ is required to compute the norm of co-vectors). Given an (invertible) metric tensor allows identifying vectors and co-vectors via the so-called musical isomorphisms. In Euclidean geometry the metric tensor is represented by a unity matrix at each point such that vectors and co-vector are represented by the same numerical values and thus usually not considered as distinct objects. However, treatment of non-Euclidean spaces requires distinguishing between vectors and co-vectors.

### 2.4 Geometric Algebra

Geometric Algebra [6, 7] extends the usual vector calculus using vector addition and scalar multiplication by introducing the geometric product as the combination of the inner product and the exterior (Grassmann) product. For two vectors $u, v \in V$ of a vector space $V$ with metric $g : V \times V \to \mathbb{R}$ the geometric product $uv$ (by convention denoted without explicit operator symbol) is given by $uv = g(u, v) + u \wedge v$. The geometric product provides useful properties such as being associative and invertible, whereas it is not commutative in general. It is closed within the space of multi-vectors, which is formed by linear combinations of $k$-vectors:

$$\Lambda(\mathcal{V}) = \bigoplus_{k=0}^{n} \Lambda^k(\mathcal{V}) \tag{6}$$

where $n$ is the dimension of the underlying vector space $V$. The dimensionality of $\Lambda(\mathcal{V})$ is $2^n$.

The Geometric Algebra (GA) requires a metric given on the underlying vector space, in contrast to the Grassmann algebra which is independent from any metric. Given the concept of a "norm" through the metric, GA allows to identify $k$-vectors with $n - k$ vectors using the hodge-star operator $\star : \Lambda^k(\mathcal{V}) \to \Lambda^{n-k}(\mathcal{V})$ since the dimensionality of $\Lambda^k(\mathcal{V})$ and $\Lambda^{n-k}(\mathcal{V})$ is equal.

### 2.5 Geometric Algebra on the Tangent Space

Given a manifold $M$ with tangent space $T_p(M)$ and co-tangent space $T_p^*(M)$ we can apply the Grassmann and Geometric algebra on each of these vector spaces. The multi-vector space constructed from tangential vectors is then $\Lambda\big(T_p(M)\big)$, the multi-vector space of co-vectors $\Lambda\big(T_p^*(M)\big)$. The multi-vector space of tangential vectors hereby corresponds to "multiple directions", such as a bi-vector defining an (oriented) area. In contrast, co-vectors describe a direction that is "not to be used", as they represent a scalar function on just this direction. A co-vector can be seen as a "cut out" function which removes a direction from the $n$-dimensional volume. For instance, in three dimensions a co-vector "cuts off" one direction, thus leaving the plane defined by the two remaining directions. Thus, a bi-co-vector is an operation that "cuts off" two directions.

A bi-co-vector corresponds to the subspace of an $n$-dimensional hyperspace where a plane is "cut out". In three dimensions these visualizations overlap: both, a bi-tangential vector and a co-vector correspond to a plane, and both a tangential vector and a bi-co-vector correspond to an one-dimensional direction ("arrow"). In four dimensions, these visuals are more distinct

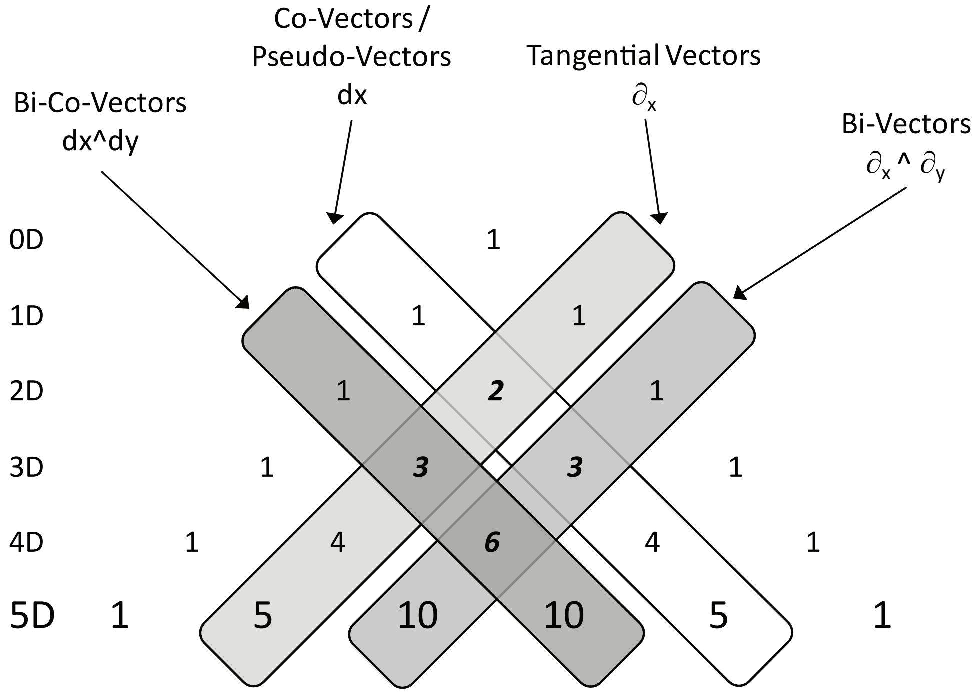


Figure 2: Pascal's triangle depicting tangential vectors, bi-vectors, co-vectors and bi-covectors for the Grassmann algebra of tangential and co-tangential spaces. In three dimensions there are many ambiguities where all vectorial objects are of the same dimensionality 3. In four dimensions there are less ambiguities , but only in 5D all vector types become unambiguous.

but still overlap: a co-vector corresponds to a three-dimensional volume, but a bi-tangential vector is represented by a plane similar to a bi-co-vector, since cutting out a 2D plane from four-dimensional space yields a 2D plane again. Only in higher dimensions these graphical representations become unique, as indicated by Fig. 2. In any case a co-vector and a pseudo-vector will have the same appearance as an $n-1$ dimensional hyperspace, same as a tangential vector corresponds to an pseudo-co-vector: The difference between a co-vector and a pseudo-vector however is their orientation: a co-vector does not provide any orientation information, it is just a "cut-off" operation. In contrast, a pseudo-vector is constructed from alternating products of tangential vectors with relevance to its ordering. In 3D, we can therefore visualize a co-vector as non-oriented planar area element and a bi-vector as an oriented planar area element.

## 3 GEOMETRIC ALGEBRA OF COLOR SPACE

### 3.1 Three-dimensional Color Space - RGB

Colors form a vector space same as spatial dimensions due to the linear properties[1] of the Maxwell equations, allowing to superpose electromagnetic waves, i.e. light. Colors in particular form a three-dimensional space due to the biophysical properties of the human eye being sensitive to three fundamental colors. There are many ways to represent a color space such as RGB, HSV, CIE etc..; a metric given on a color space may for instance be used to determine color similarities when comparing images [3].

[1]In contrast, gravitational waves cannot be superposed due to the non-linearity of the Einstein equations.

A most simple color space is RGB, using the red, green and blue primary colors to construct other colors, similar to the x,y,z coordinates in Euclidean vector calculus. It is common to associate xyz coordinates with RGB colors for illustration purposes in computer graphics. The RGB color space describes the additive color system as it represents light being added to darkness, for instance, relevant to LED displays or computer/TV monitors. Complementary to RGB is the CMY color space describing the subtractive color system where color filtering is applied to a white background, such as in painting or printing colors on a white sheet of paper. The primary colors of the CMY color space are cyan, magenta and yellow, which are complementary to red, green and blue in the additive color system.

When considering the notion of tangential and co-tangential vectors in differential geometry, it seems quite fitting to associate RGB colors with tangential vectors and CMY colors with co-vectors: RGB colors "add" directional information, like tangential vectors, whereas CMY colors "subtract" directional information, like co-vectors. The intrinsic duality between vectors and co-vectors corresponds to the complementarity of RGB and CMY colors. We may denote the basis vectors of the RGB space as $\{\partial_r, \partial_g, \partial_b\}$ and of the CMY space as $\{\partial_c, \partial_m, \partial_y\}$ to support this interpretation. Duality between RGB and CMY then means in this notation:

$$\star\,\partial_r = dr \equiv \partial_c \tag{7}$$
$$\star\partial_g = dg \equiv \partial_m \tag{8}$$
$$\star\partial_b = db \equiv \partial_y \tag{9}$$
$$\star\partial_c = dc \equiv \partial_r \tag{10}$$
$$\star\partial_m = dm \equiv \partial_g \tag{11}$$
$$\star\partial_y = dy \equiv \partial_b \tag{12}$$

The exterior product of color vectors then corresponds to the mixing of colors in this interpretation, allowing for additive and subtractive mixing. For instance, mixing red and green light yields yellow light, described as $\partial_r \wedge \partial_g$, a bivector in the additive color space. Yellow ($\partial_y$) in the CMY system can be seen as removing blue from white and is thus identified with a co-vector in the RGB system, $db \equiv \partial_y$. This co-vector is identified with the dual of the blue color, i.e. $\star\partial_b \cong \partial_r \wedge \partial_g$. This identification is the same as relating the cross product "$\times$" of vectors with a "normal" vector in Euclidean vector calculus $\partial_z \cong \partial_x \times \partial_y = \star(\partial_x \times \partial_y)$. Identifying the same "colors" in the additive (RGB) color systems with colors in the subtractive (CMY) color system is the same as identifying vectors with co-vectors and bi-vectors. This is only possible within a three-dimensional vector space.

### 3.2 Four-dimensional Color Space - RGBA

While human vision is limited to only three primary colors, in computer graphics color is frequently described using four components. The fourth component is transparency. Considering transparency as an additional property of a color space allows to consider a four-dimensional space. The property complementary to transparency is opacity, or "blackness", which may be expressed via the CMYK color model where "k" stands for an additional blackness component, similar to opacity of the overall color. Starting from the four-dimensional RGBA color space with four basis vectors $\{\partial_r, \partial_g, \partial_b, \partial_a\}$ we may construct six color bi-vectors

$$
\begin{aligned}
\partial_r \wedge \partial_a &\rightarrow \text{transparent red} \\
\partial_g \wedge \partial_a &\rightarrow \text{transparent green} \\
\partial_b \wedge \partial_a &\rightarrow \text{transparent blue} \\
\partial_r \wedge \partial_g &\rightarrow \text{yellow (non-transparent)} \\
\partial_g \wedge \partial_b &\rightarrow \text{cyan (non-transparent)} \\
\partial_b \wedge \partial_r &\rightarrow \text{magenta (non-transparent)}
\end{aligned}
$$

four three-vectors

$$
\begin{aligned}
\partial_r \wedge \partial_g \wedge \partial_a &\rightarrow \text{transparent yellow} \\
\partial_g \wedge \partial_b \wedge \partial_a &\rightarrow \text{transparent cyan} \\
\partial_b \wedge \partial_r \wedge \partial_a &\rightarrow \text{transparent magenta} \\
\partial_r \wedge \partial_g \wedge \partial_b &\rightarrow \text{gray (non-transparent)}
\end{aligned}
$$

and last not least one pseudo-scalar

$$
\partial_r \wedge \partial_g \wedge \partial_b \wedge \partial_a \rightarrow \text{transparent white} \quad .
$$

These various wedge products allow to express all possible combinations to be built from RGBA. Using the relationships eqn.(7), (8) and (9) we can express mixed colors via their dual counterparts from the CMYK space, e.g.

$$
\partial_r \wedge \partial_g = dc \wedge dm = \star(\partial_c \wedge \partial_m) \tag{13}
$$

where we can interpret using the co-vector as indicating a color that "must not" be used. The star-operator hereby plays the role of a "not" operator, i.e. the indicated color blue in the subtractive color model, expressed as $\partial_c \wedge \partial_m$ (mixing cyan and magenta gives blue) must not be used for printing the additive color yellow $\partial_r \wedge \partial_g$. Using vectors, the $\wedge$ -operator means "both", using co-vectors, the $\wedge$ operator means "neither". Thus we can identify RGBA colors with their dual counterparts from the CMYK space by considering "color that should not be printed" on a white canvas, whereby "transparency" corresponds to some "light" or "bright" color in this case:

$$
\begin{aligned}
\partial_r \wedge \partial_a = \star(\partial_c \wedge \partial_k) &\rightarrow \text{light red means printing neither black nor cyan} \\
\partial_g \wedge \partial_a = \star(\partial_m \wedge \partial_k) &\rightarrow \text{light green means printing neither black nor magenta} \\
\partial_b \wedge \partial_a = \star(\partial_y \wedge \partial_k) &\rightarrow \text{light blue means printing neither black nor yellow} \\
\partial_r \wedge \partial_g = \star(\partial_c \wedge \partial_m) &\rightarrow \text{intense yellow means printing neither cyan nor magenta} \\
\partial_g \wedge \partial_b = \star(\partial_m \wedge \partial_y) &\rightarrow \text{intense cyan means printing neither magenta nor yellow} \\
\partial_b \wedge \partial_r = \star(\partial_y \wedge \partial_c) &\rightarrow \text{intense magenta means printing neither yellow nor cyan}
\end{aligned}
$$

Expressing a rule "to print colors" means that all CMYK colors are printed except those specified to not be printed. The four tri-vector colors relate to the RGBA/CMYK color space as follows:

$$\begin{aligned}
\partial_r \wedge \partial_g \wedge \partial_b = da = dc \wedge dm \wedge dy = \star dk = \partial_k &\quad\rightarrow\quad \text{gray - no white - print black} \\
\partial_r \wedge \partial_g \wedge \partial_a = db = dc \wedge dm \wedge dk = \star dy = \partial_y &\quad\rightarrow\quad \text{light yellow - no blue - print yellow} \\
\partial_g \wedge \partial_b \wedge \partial_a = dr = dm \wedge dy \wedge dk = \star dc = \partial_c &\quad\rightarrow\quad \text{light cyan - no red - print cyan} \\
\partial_b \wedge \partial_r \wedge \partial_a = dg = dy \wedge dc \wedge dk = \star dm = \partial_m &\quad\rightarrow\quad \text{light magenta - no green - print magenta}
\end{aligned}$$

The relationships between RGBA color vectors and their CMYK counterparts may therefore be interpreted as "rules" on how to print additive colors using the subtractive color model. The inverse rules, not shown here explicitly, can be interpreted as rules how to produce subtractive colors using a light projector. Arbitrary colors are created via linear combinations of these base vectors. The pseudo-scalars represents white and black:

$$\begin{aligned}
\partial_r \wedge \partial_g \wedge \partial_b \wedge \partial_a = dc \wedge dm \wedge dy \wedge dk &\quad\rightarrow\quad \text{white - do not print any color} \\
dr \wedge dg \wedge db \wedge da = \partial_c \wedge \partial_m \wedge \partial_y \wedge \partial_k &\quad\rightarrow\quad \text{black - do print all colors}
\end{aligned}$$

In this four-dimensional RGBA color space, vectors (i.e. primary colors in the additive color system: red, green, blue, transparent) can no longer be identified with bi-vectors (light red, light green, light blue, intense yellow, intense cyan, intense magenta) as it was possible in the three-dimensional RGBA color space. We can however identify tri-vectors with co-vectors, since using three colors means not using the fourth color, whereby "color" hereby also includes the transparency/black channel.

This illustration of the base vectors of four-dimensional space is similar to the case of space-time algebra (STA) [5] where we have three spacelike and three timelike, i.e. six, bi-vectors. Using these color vectors, the transparency (alpha channel, black channel for printing) playes the same role as time in STA, hereby distinguishing "bright" and "dark" colors.

The $2^4 = 16$ colors that can be produced by a system of four light emitting diodes (LEDs) (red, green, blue and white) are summarized in the following table, where color vectors indicate the LEDs to be switched on and co-vectors indicate LEDs that must not be used:

| RGBA vector<br>*LED's to use* | RGBA covector<br>*LED's not to use* | color |
|---|---|---|
| $0$ | $dr \wedge dg \wedge db \wedge da$ | black |
| $\partial_r$ | $dg \wedge db \wedge da$ | dark red |
| $\partial_g$ | $dr \wedge db \wedge da$ | dark green |
| $\partial_b$ | $dr \wedge dg \wedge da$ | dark blue |
| $\partial_a$ | $dr \wedge dg \wedge db$ | transparency / brightness |
| $\partial_r \wedge \partial_a$ | $dg \wedge db$ | bright red |
| $\partial_g \wedge \partial_a$ | $dr \wedge db$ | bright green |
| $\partial_b \wedge \partial_a$ | $dr \wedge dg$ | bright blue |
| $\partial_r \wedge \partial_g$ | $db \wedge da$ | dark yellow |
| $\partial_g \wedge \partial_b$ | $dr \wedge da$ | dark cyan |
| $\partial_b \wedge \partial_r$ | $dg \wedge da$ | dark magenta |
| $\partial_r \wedge \partial_g \wedge \partial_b$ | $da$ | gray |
| $\partial_r \wedge \partial_g \wedge \partial_a$ | $db$ | bright yellow |
| $\partial_g \wedge \partial_b \wedge \partial_a$ | $dr$ | bright cyan |
| $\partial_b \wedge \partial_r \wedge \partial_a$ | $dg$ | bright magenta |
| $\partial_r \wedge \partial_g \wedge \partial_b \wedge \partial_a$ | $0$ | white |

It should be pointed out that both $\partial_a$ and $\partial_r \wedge \partial_g \wedge \partial_b$ represent gray; this corresponds to the two alternative ways of a CMYK printer to produce a gray color: either using all of CMY inks, or using just the black ink. In theory, both methods would yield gray. In practice, there will be slight deviations in the gray tone. The ambiguity among these two approaches may therefore be resolved by assuming that $\partial_a$ yields an overall intensity different from $\partial_r \wedge \partial_g \wedge \partial_b$, for instance "bright gray" vs. "dark gray".

When identifying bright and dark colors, i.e. ignoring the alpha channel/intensity, we arrive at the three-dimensional color space as before where vectors and co-vectors can be identified with each other. This, again, relates to identifying vectors and bi-vectors in 3D ("bright red" $\approx$ "dark red").

### 3.3 Five-dimensional Color Space - RGBAT/CMYKU

The four-dimensional RGBA color space could easily be extended to a five-dimensional scheme by introducing different levels of gray, introducing infra-red or ultra-violet, but this would become rather confusing and less intuitive since these extensions are not easily accessible to the perception of the human eye and thus our experiences. A more suitable approach is to add texturing as a fifth parameter to the color space, now considering not just single dots of colors, but some spatially extend elements that may or may not provide some additional pattern. Denoting this texturing property as a "t" coordinate, we have the following basis vectors for a five-dimensional RGBAT color space:

$$\begin{aligned}
\partial_r &\rightarrow \text{red} \\
\partial_g &\rightarrow \text{green} \\
\partial_b &\rightarrow \text{blue} \\
\partial_a &\rightarrow \text{brightness} \\
\partial_t &\rightarrow \text{textured}
\end{aligned}$$

These five base vectors produce $2^5 = 32$ combinations for colored patterns, encompassing 10 bi-vectors, 10 tri-vectors, 5 four-vectors and one pseudoscalar. Instead of listing each of these cases, the main purpose of considering a five-dimensional color space is to show that each of these objects is different in 5D as pointed out in Fig. 2, and none of them can be identified with each other any more like in 3D or 4D.

In 3D we could say "red is not cyan" ($\partial_r = dc$), "red light means using neither green and blue light" ($\partial_r = \star(\partial_g \wedge \partial_b)$), "printing red means mixing magenta and yellow colors" ($\partial_r = \star(\partial_m \wedge \partial_y)$), allowing for alternative ways to produce the same color in RGB space, i.e. identifying vectors (primary RGB colors) with bi-vectors (specifying which lights not to use) and co-vectors (specifying which inks to use for printing): pure or mixed color in the three-dimensional scheme is a primary color in either the RGB ("tangential") or CMY ("co-tangential") space. Thus for each color we have the option to uniquely specify it by just giving one colorization vector (i.e. any of "red","green","blue", "magenta","cyan" or "yellow").

In 4D, any primary color ("dark red") can only be expressed by three complementary properties (e.g. $\partial_r = dg \wedge db \wedge da$). Any mixed color can only be expressed by a mix of complementary colors as well (e.g. bright red $\partial_r \wedge \partial_a = dg \wedge db$), but never as a primary color in either RGBA or CMYK. This corresponds to identifying vectors with tri-covectors and bi-vectors with bi-co-vectors in 4D, e.g.

| | | | |
|---|---|---|---|
| dark red | $\partial_r$ | $dg \wedge db \wedge da$ | vector / tri-co-vector |
| bright red | $\partial_r \wedge \partial_a$ | $dg \wedge db$ | bi-vector / bi-co-vector |
| bright magenta | $\partial_r \wedge \partial_b \wedge \partial_a$ | $dg$ | tri-vector / co-vector |

Thus in 4D space we have a unique choice to express primary colors, but alternative options to express mixed colors if we want to specify only two color properties from either RGBA or CMYK.

In 5D, we can still express each $k$-vector via its dual $n-k$ co-vector (adding colors/textures vs. removing colors/textures), but the descriptions will no longer be equivalent. For instance, a "dark red textured" pattern is the bi-vector $\partial_r \wedge \partial_t$, "bright red textured" is the tri-vector $\partial_r \wedge \partial_a \wedge \partial_t$ and "bright magenta textured" is the four-vector $\partial_r \wedge \partial_b \wedge \partial_a \wedge \partial_t$, see Fig. 3.

| | | | |
|---|---|---|---|
| black | 0 | $dr \wedge dg \wedge db \wedge da \wedge dr$ | scalar |
| dark red | $\partial_r$ | $dg \wedge db \wedge da \wedge dr$ | vector |
| dark red textured | $\partial_r \wedge \partial_t$ | $dg \wedge db \wedge da$ | bi-vector |
| bright red textured | $\partial_r \wedge \partial_a \wedge \partial_t$ | $dg \wedge db$ | bi-co-vector |
| bright magenta textured | $\partial_r \wedge \partial_b \wedge \partial_a \wedge \partial_t$ | $dg$ | co-vector |
| bright white textured | $\partial_r \wedge \partial_g \wedge \partial_b \wedge \partial_a \wedge \partial_t$ | 0 | pseudo-scalar |

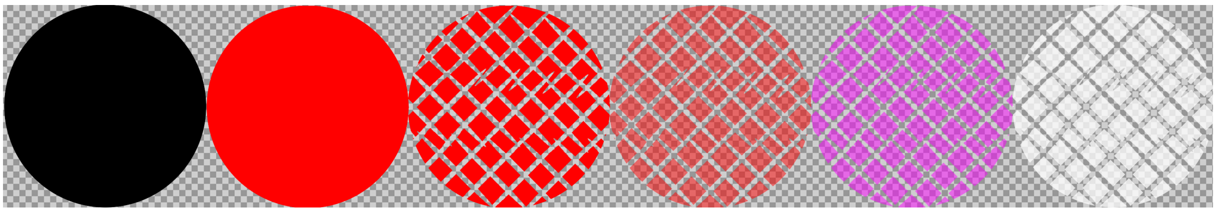

Figure 3: Six exemplary and unique 5D multi-vector elements illustrated in 5D color space using red, green, blue, alpha and texture (RGBAT). From left to right: scalar, vector, bi-vector, bi-co-vector, co-vector and pseudo-scalar.

In this 5D space we may still express any color via just two properties from either RGBAT or its complementary CMYKU ("U" meaning "untextured") space, but in contrast to the 3D and 4D space this choice will be unique now: A mixed color must either be expressed via RGBAT quantities or CMYKU quantities such that two basis vectors are sufficient for identifying the color. This property of the five-dimensional texture color space disables the ability to identify vectors with co-vectors, bi-vectors and bi-co-vectors.

## 4 SUMMARY

Considering the properties of colors is an alternative to geometry for illustrating the notion of tangential vectors and co-vectors in differential geometry in conjunction with Grassman and geometric algebra. This approach eases intuition for higher-dimensional spaces. A tangential vector can be understood as a "adding" a property to a colorization scheme, a co-vector can be understood as a rule to "remove" a property from a color. The wedge product is the join of these properties, leading to objects such as bi-vectors, co-vectors and bi-co-vectors. In 3D and 4D these objects can be described in alternative ways, only in 5D the "most easy" description in terms of a "minimal number of properties to be specified" becomes unique.

## 5 FUTURE WORK

The discussion on applying geometric algebra and differential geometry to color spaces is intended to be of primarily didactic purpose; it may lead to inspirations and further applications, for instance in image processing where the choice of color/pattern metric plays a relevant role for similarity assessments based on perception theory such as in [8, 3]. The provided discussion does not utilize the full mathematical framework of geometric algebra considering such a metric. For a meaningful interpretation of colors as vectors and bi-vectors one would also have to consider the ordering of colors ($\partial_r \wedge \partial_g = -\partial_g \wedge \partial_r$), i.e negative intensities for color values.

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